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Introduction

Selection of a rafter system for roof framing can depend on a variety of considerations, including span requirements, opportunities to provide rafter support, load magnitudes and distributions, roof pitch and aesthetics. Each method for supporting a pair of rafters has its own set of advantages and disadvantages affecting member size as well as building performance. The objective of this bulletin, which is based upon prior published work (see Ref. [1]), is to present structural analysis equations and a comparison of various rafter systems for pitched roofs such that the implications of a specific choice are clear to the designer.

Seven different rafter systems are considered: four of which are statically determinate, and three of which are indeterminate to the first degree (with a single redundant). The seven options are illustrated in Figure 1. In the figure, the symbol  represents a hinged support location that is capable of resisting both vertical and horizontal displacement of the rafter at that location. The symbol  represents a roller support capable of resisting only vertical displacement; horizontal displacement is not restrained in this case. The discussion is limited to uniform gravity load that is symmetric about the ridgeline. Unbalanced and nonuniform gravity loads require alternative analysis.

For each system, the structural analysis results include expressions for reactions, axial force, shear force, and bending moment. We have combined these results with additional practical information to provide designers with the resources to make an appropriate selection.

Since each of the systems is symmetric about the ridge, it is sufficient to consider only one of the two rafters in the pair. In the free-body diagrams that follow (Figures 2 and 4), point A identifies the eave support, point B locates the ridge, and point C is the attachment or support point that occurs at a location other than at the rafter ends. For each rafter system, the span L is taken as the horizontal projection of the distance from the eave to the ridge. Span coordinates x, x_1, and x_2 are also measured horizontally, as indicated in the figures. Uniformly distributed gravity load

w is applied along the horizontal projection of the rafter. Roof pitch (rafter slope) is expressed in terms of the angle θ relative to the horizontal.

For rafters that include support or attachment at a location other than at the rafter ends, a span factor α ($0 < \alpha < 1$) is used to express the location of the interior support or attachment point. Cantilever rafter tails are neglected. The positive directions of support reactions are shown in the accompanying free-body diagrams, and the usual sign conventions from mechanics of materials for axial force, shear force and bending moment apply.

Wind loading is beyond the scope of this work, but it also plays an important role in roof-system design. However, under most circumstances gravity load will control the size of the rafters.

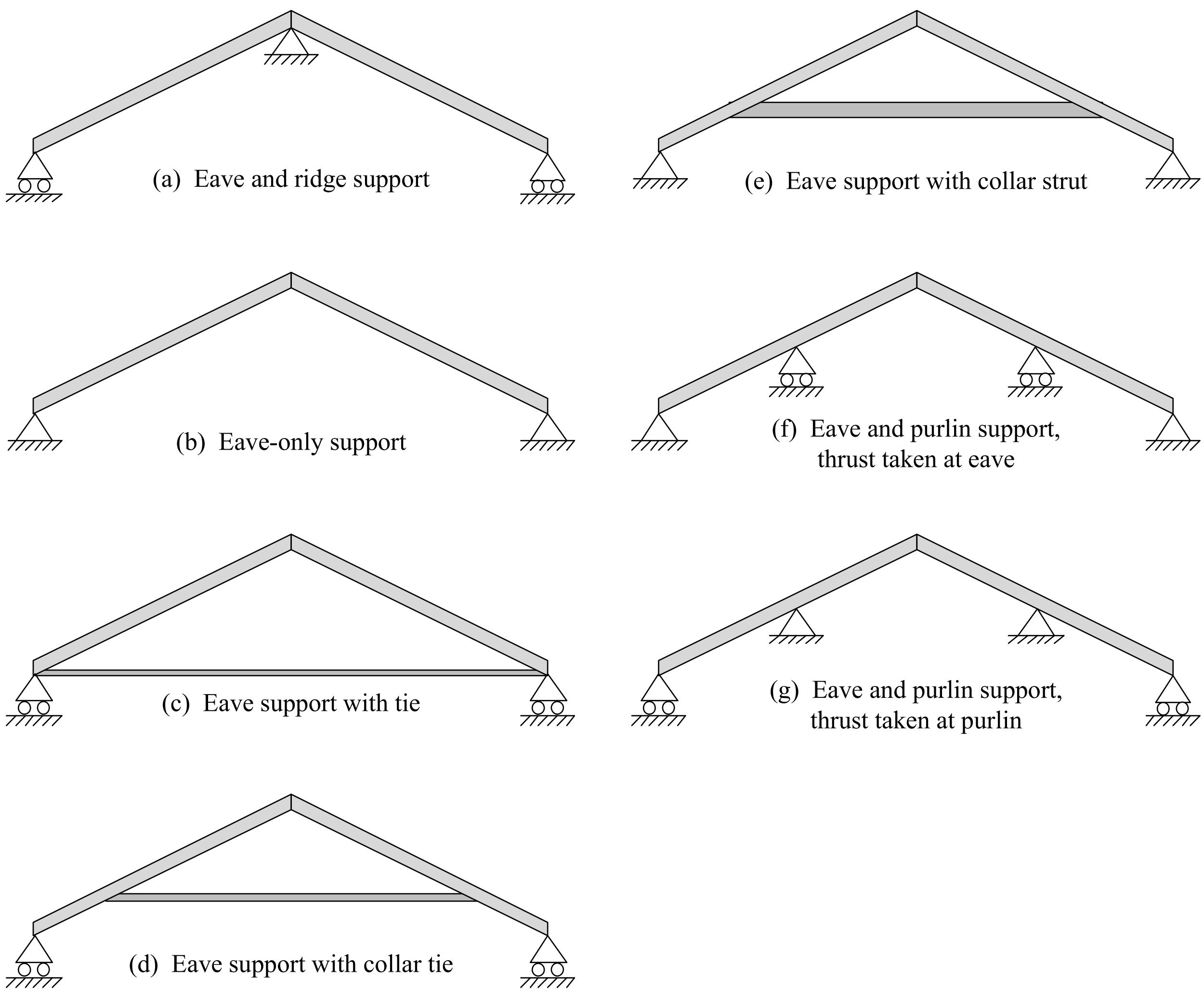

Figure 1 – Rafter support options

Statically Determinate Options

The first four rafter options are statically determinate. Hence their analyses rely solely on the principles of static equilibrium. Figure 2 shows free-body diagrams for these options.

Eave and Ridge Support

The simplest system, but not necessarily the most common one, involves a pair of simply supported rafters, supported at their ends by the building eaves and a ridge beam (Figures 1(a) and 2(a)). In this configuration, both rafters of the pair respond to load without interaction at the ridge[1]. This configuration yields generous, unobstructed headroom below the plane of the roof. The eave plates and ridge beam provide vertical support to the rafters and no outward thrust is delivered to the eave support.

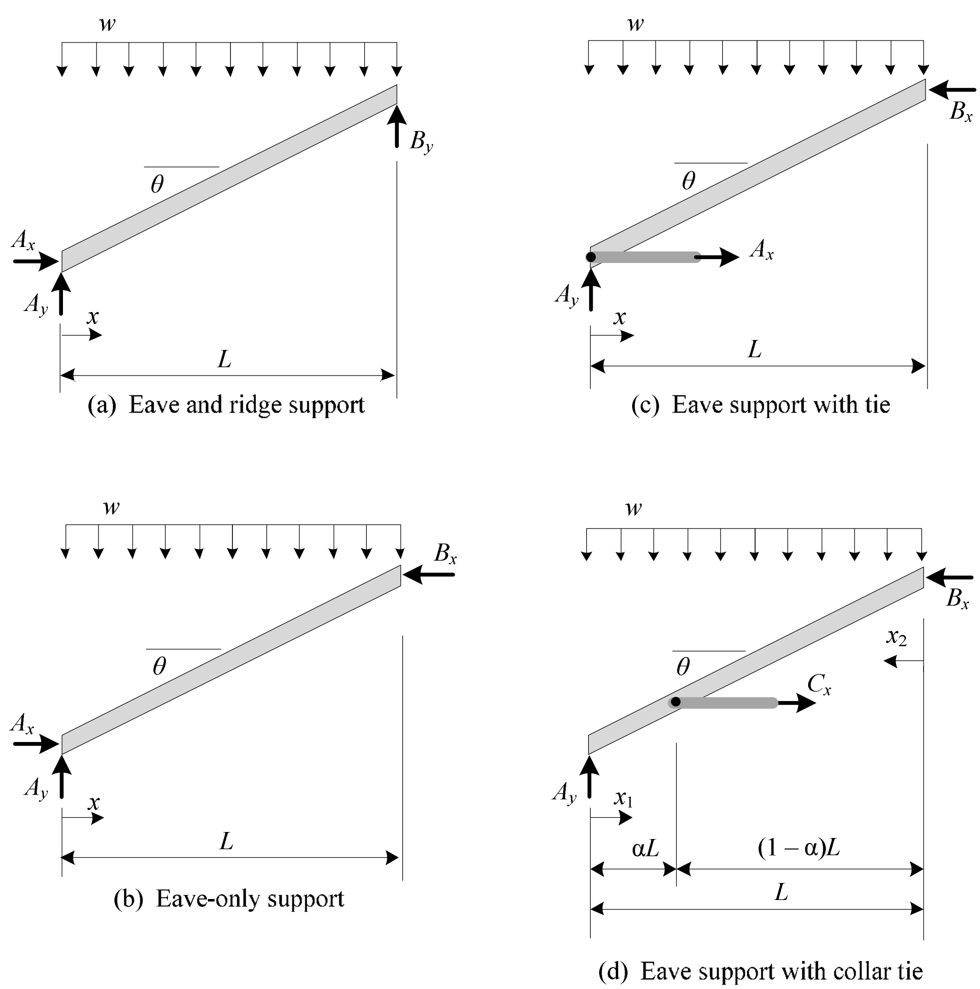

Figure 2 – Free-body diagrams, statically determinate options

[1] All subsequently discussed support options involve interaction between the rafters at the ridge.

The analysis model is statically determinate with reactions shown in the free-body diagram (Figure 2(a)). Support reactions A_x, A_y, B_y, axial force N, shear force V, and bending moment M are given by Equations (1).

$$A_x = 0, \quad A_y = B_y = \frac{wL}{2}$$

$$N = w\left(x - \frac{L}{2}\right)\sin\theta\,, \quad V = w\left(\frac{L}{2} - x\right)\cos\theta\ , \quad M = \frac{wx}{2}(L - x) \qquad \text{Eq. (1)}$$

Because the rafters carry their loads primarily through bending, the roof span will be limited by the depth of the rafters or the length of the available framing material. In addition, the ridge beam must be designed to carry fully 50% of the total load on the roof, which might lead to a prohibitively deep member or to multiple support posts along the ridgeline.

A benefit to the eave and ridge supported rafter system is that each rafter in the pair is completely independent of its mate. This independence means that rafters that can be installed individually and are not prone to progressive collapse. In the following three statically determinate options, failure of one rafter in the pair would result in loss of mutual support at the ridge and hence collapse of the other rafter as well. Alternative load paths, such as those provided by continuous roof sheathing or longitudinal purlins may serve to prevent catastrophic roof failure.

Eave-only Support

An alternative to a deep ridge beam or multiple ridgeline supports is to eliminate the ridge beam and allow the rafters to lean against each other (Figure 1(b)). The analysis model remains statically determinate, with reactions and internal actions given by Equations (2).

$$A_x = B_x = \frac{wL}{2\tan\theta}, \qquad A_y = wL$$

$$N = w\left[x - L\left(1 + \frac{1}{2\tan^2\theta}\right)\right]\sin\theta\,, \quad V = w\left(\frac{L}{2} - x\right)\cos\theta\ , \quad M = \frac{wx}{2}(L - x) \qquad \text{Eq. (2)}$$

The shear force and bending moment in the rafters for this option are identical to those for the simply supported option. However, axial force is substantially larger, suggesting that rafter design might be governed by combined compression and bending interaction.

Another effect of eliminating the ridge support is that all gravity load is carried into the vertical support at the eave. Likely more significant is the outward thrust on the wall plates at the eave caused by the axial compression in the rafters. To resist the thrust, the plates would need to be heaver (wider) to carry the thrust into the building's end walls, or some form of transverse support, such as one or more shear walls, for the eave must be used.

To facilitate rafter installation, a thin ridge board may be utilized, allowing rafters to be installed one at a time, rather than concurrently as mating pairs. This alignment ridge is not intended to serve as a structural component. Such an application leads to an eave-only support condition.

Eave Support with Tie

The thrust on the walls from the eave-only support option is efficiently resisted with a tension tie at eave level. The addition of this eave tie essentially converts the rafter pair into a simple truss configuration with superimposed rafter-chord bending. The analysis model (Figure 1(c)) and free-body diagram (Figure 2(c)) are essentially the same as those for eave-only support, and the analysis results are given by Equations (2).

The tension tie provides an efficient means to resist rafter thrust, but it also results in loss of headroom under the roof, often an unacceptable outcome, such as when vaulted ceilings or exposed roof framing are desired. The combined compression and bending in the rafters remains a design consideration, as does the tension connection between the rafter and the tie at the eave.

Eave Support with Collar Tie

Some of the highly prized headroom can be recovered by replacing the eave-level tension tie with a collar tie (Figure 1(d)) located up from the eave. The attachment point for the collar tie (Figure 2(d)) lies at a horizontal distance αL from the eave support. The tension member in this case is often called a collar tie, irrespective of how high along the rafter it is joined. The system remains statically determinate and support reactions are given by Equations (3).

$$A_y = wL, \quad B_x = C_x = \frac{wL}{2(1-\alpha)\tan\theta} \qquad \text{Eq. (3)}$$

The collar tie causes discontinuities in the internal-action equations at point C. Left of point C, for $0 \le x_1 \le \alpha L$, the internal actions are given by Equations (4a). Equations (4b) give internal actions to the right of point C and are valid for $0 \le x_2 \le (1-\alpha)L$.

$$N_1 = w(x_1 - L)\sin\theta, \qquad V_1 = w(L - x_1)\cos\theta, \qquad M_1 = wx_1\left(L - \frac{x_1}{2}\right) \qquad \text{Eq. (4a)}$$

$$N_2 = -w\left(x_2 + \frac{L}{2(1-\alpha)\tan^2\theta}\right)\sin\theta, \quad V_2 = w\left(x_2 - \frac{L}{2(1-\alpha)}\right)\cos\theta,$$

$$M_2 = \frac{wx_2}{2}\left(\frac{L}{(1-\alpha)} - x_2\right) \qquad \text{Eq. (4b)}$$

To be effective in restoring headroom, the collar tie must be as high on the rafter as possible. However, bending moment in the rafter increases rapidly as the span factor α increases. Figure 3 illustrates the bending moment diagrams for the rafter for a range of values of α. The moment diagrams are normalized such that the maximum moment in the rafter is 1.0 when the collar tie is attached at the eave ($\alpha = 0$).

The maximum moment in the rafter occurs at attachment point *C* when the span factor $\alpha \geq \alpha_0$, where $\alpha_0 = 1 - 1/\sqrt{2} = 0.29289$. When $\alpha = \alpha_0$, the moment at point *C* is exactly twice the maximum rafter moment for the baseline case (when $\alpha = 0$). Hence when $\alpha \geq \alpha_0$, rafter design will be controlled by bending of the rafter at point *C*. If the connection of the collar tie to the rafter at *C* reduces the cross section of the rafter, the rafter's ability to resist bending is further challenged.

Axial compression in the rafter segment between the collar tie attachment point and the ridge might compound the design challenge. Also, raising the collar tie increases the tension in the tie compared to the eave-tied option, further complicating the rafter-to-tie connection.

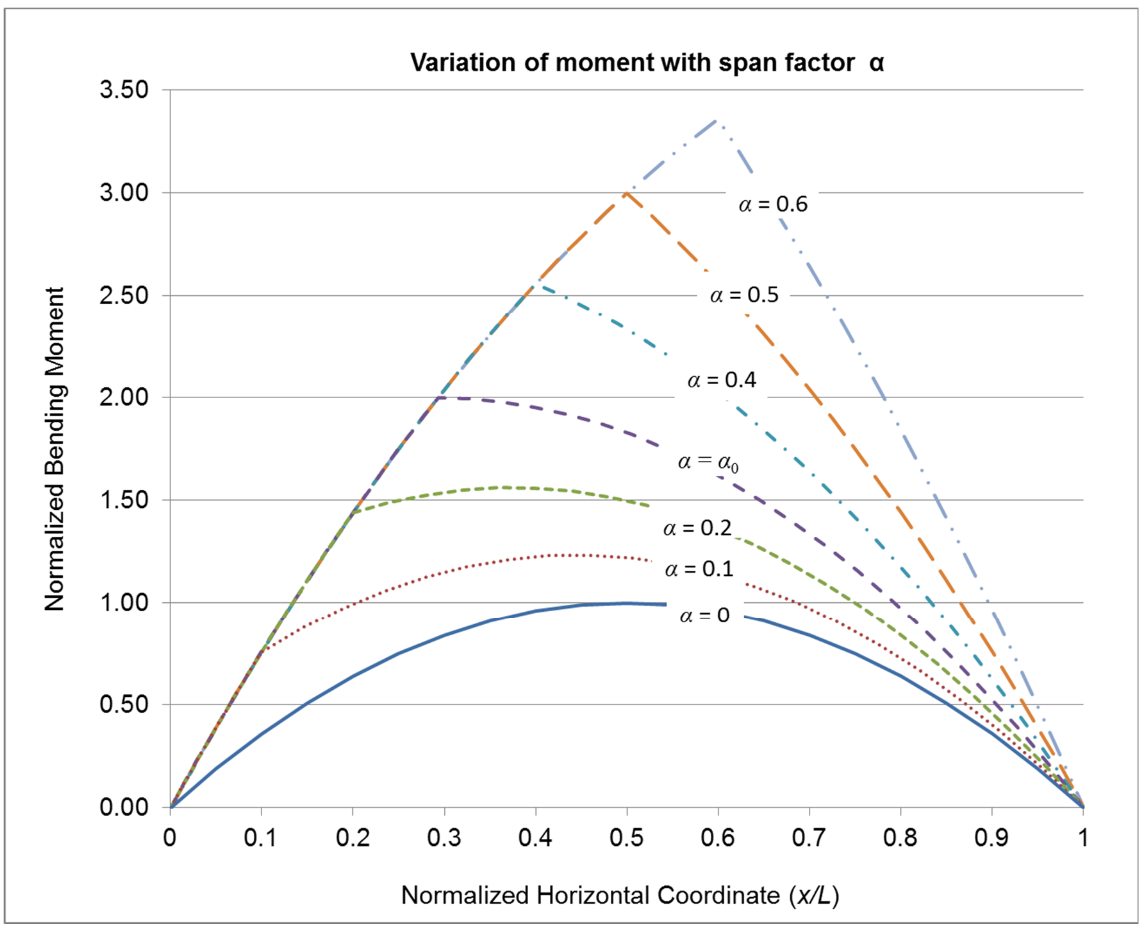

Figure 3 – Bending moment parameter study, eave supported rafter with collar tie.

Statically Indeterminate Options

The remaining support options shown in Figures 1 (e) - (g) are statically indeterminate. Figure 4 contains free-body diagrams for these options. Structural analysis of these options requires use of one of several methods available for indeterminate systems with a single redundant. In this case, Castigliano's theorem on deflections (see Ref. [2]) with enforcement of displacement compatibility at the redundant reaction was utilized. Details of the analysis method are not presented here, as they are reasonably well known. In that analysis, only the strain energy due to bending was considered; axial and shear deformation of the rafter and collar strut were not included.

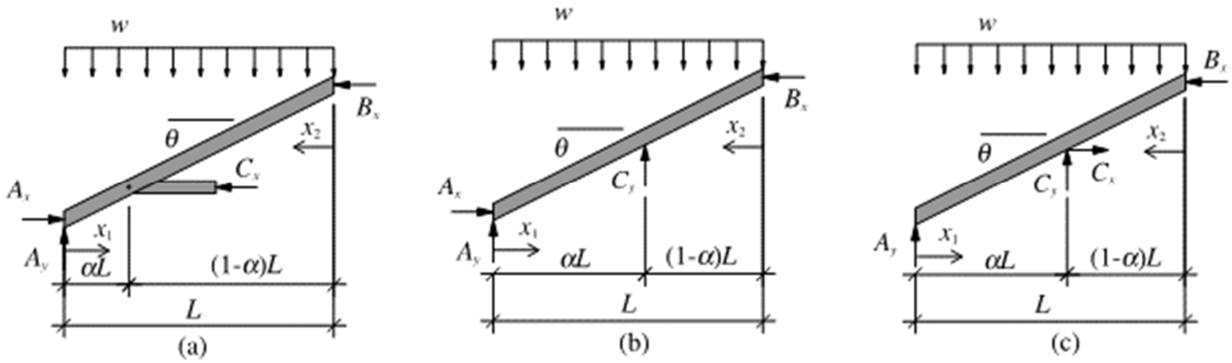

Figure 4 – Free-body diagrams, statically indeterminate options

Eave Support with Collar Strut

Those who study, preserve and restore old timber buildings have occasionally observed that in some situations, the collar tie, which is often also a timber member, actually carries compression and serves to resist sag of the rafters. Such behavior, whether intended or not, is a consequence of different support conditions than those illustrated in Figure 1(d). The support conditions represented in Figure 1(e), in which the eave support is capable of resisting thrust, render the system statically indeterminate, with the horizontal member acting as a compression strut rather than as a tension tie. Hence, the distinction between a collar tie and a collar strut lies in the ability of the eave plate to resist thrust, not in how high up the rafter the collar is placed.

For the free-body diagram in Figure 4(a), the support reactions are given in Equations (5).

$$A_x = \frac{wL}{8\tan\theta}\left(\frac{-\alpha^2+5\alpha+1}{\alpha}\right), \quad A_y = wL,$$

$$B_x = \frac{wL}{8\tan\theta}\left(\frac{\alpha^2-5\alpha+3}{1-\alpha}\right), \qquad C_x = \frac{wL}{8\tan\theta}\left(\frac{-\alpha^2+\alpha+1}{\alpha(1-\alpha)}\right) \qquad \text{Eq. (5)}$$

As with the collar tie, the collar strut causes discontinuities in the axial force, shear force and bending moment equations at the attachment point. Left of point *C*, for $0 \leq x_1 \leq \alpha L$, the internal actions are given by Equations (6a). Equations (6b) give the internal actions to the right of point *C* and are valid for $0 \leq x_2 \leq (1-\alpha)L$.

$$N_1 = w\left[x_1 - L\left(1 - \frac{\alpha^2 - 5\alpha - 1}{8\alpha \tan^2\theta}\right)\right]\sin\theta,\quad V_1 = (w\cos\theta)\left[\frac{L(\alpha^2 + 3\alpha - 1)}{8\alpha} - x_1\right]$$

$$M_1 = wx_1\left[\frac{L(\alpha^2 + 3\alpha - 1)}{8\alpha} - \frac{x_1}{2}\right] \qquad \text{Eq. (6a)}$$

$$N_2 = -w\left[x_2 + L\left(\frac{\alpha^2 - 5\alpha + 3}{8(1-\alpha)\tan^2\theta}\right)\right]\sin\theta,\quad V_2 = (w\cos\theta)\left[x_2 - \frac{L(\alpha^2 - 5\alpha + 3)}{8(1-\alpha)}\right]$$

$$M_2 = wx_2\left[\frac{L(\alpha^2 - 5\alpha + 3)}{8(1-\alpha)} - \frac{x_2}{2}\right] \qquad \text{Eq. (6b)}$$

The equations for bending moment in the collar-strut supported rafter are identical to those for a uniformly loaded, pin-supported, two-span, continuous beam with overall length *L* and interior support at αL (see Ref. [3]); the respective equations for rafter shear force differ by a factor of ($\cos\theta$).

The optimal location for the collar strut attachment, corresponding to minimum bending moment in the rafter, is at $\alpha = 0.5$. The moment diagrams in Figure 5 illustrate the effect of the span factor α. The diagrams are normalized such that the maximum moment in the rafter at attachment point *C* is -1.0 for $\alpha = 0.5$. Note that the compressive axial force in the collar strut mandates significant and corresponding thrust resistance at the eave.

Normally statically determinate systems, containing redundant load paths, are regarded as relatively safe from progressive collapse. However, the collar-strut system presented here does not possess that safety. Failure of a single rafter in the pair would result in loss of lateral support at the ridge as well as collar-strut anchorage of the remaining rafter. Supplemental load paths, such as continuous sheathing would be needed to prevent collapse. However, the following two rafter support options do indeed possess well-defined redundant load paths that will help resist catastrophic failure of the roof system.

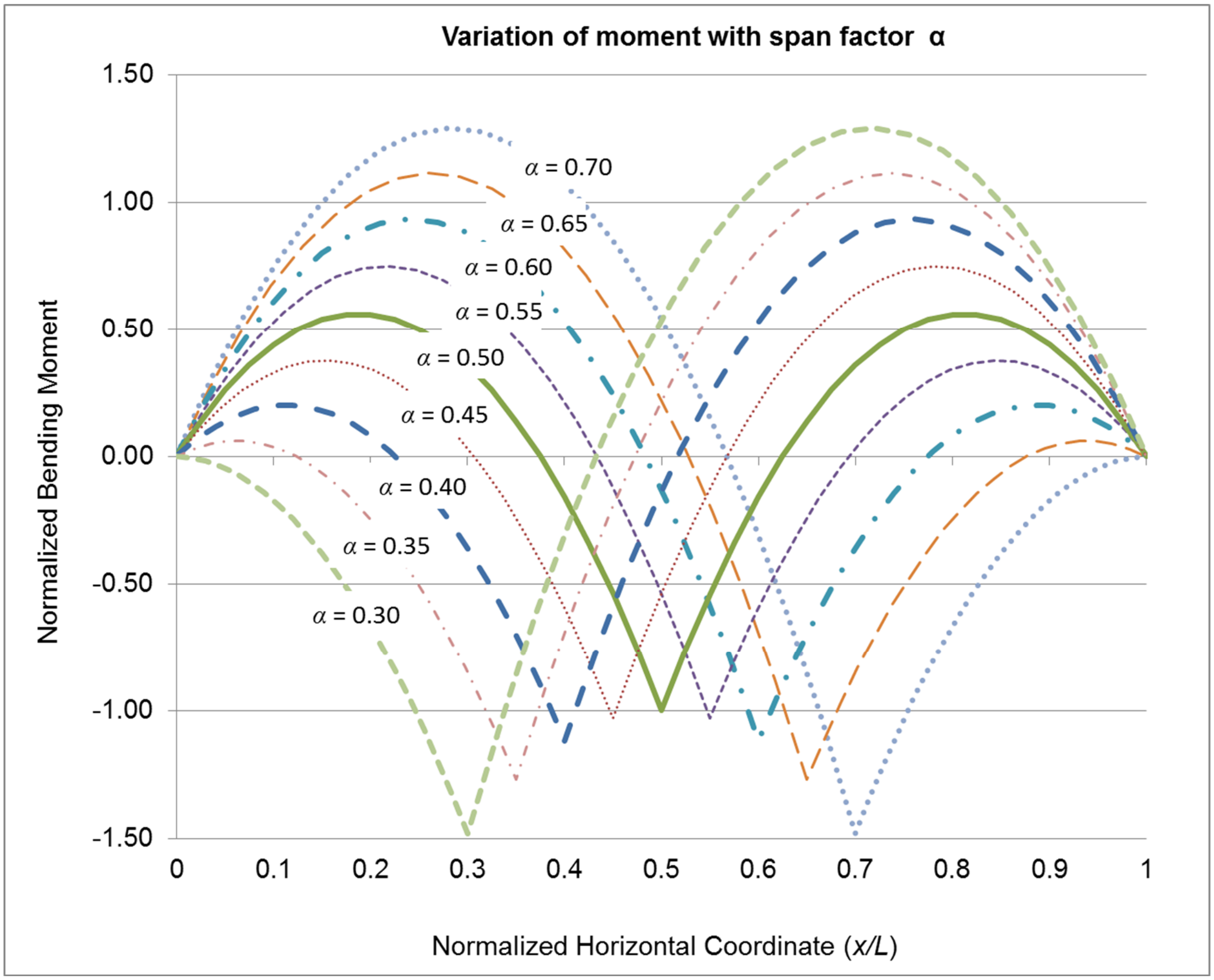

Figure 5 – Bending moment parameter study, eave supported rafter with collar strut.

Eave and Purlin Support

The collar strut serves to reduce bending moment in the rafter by introducing an interior support, producing behavior similar to that of a two-span continuous beam. An alternative to using a collar strut to reduce bending moments in the rafters involves use of vertical (rather than horizontal) support along the interior of the rafter at point *C*. For common rafters, such support is often provided by principal purlins oriented parallel to the ridgeline. For a principal rafter, the interior support generally takes the form of a vertical post or canted strut as part of a bent in the structural frame. For the remaining two options, we assume that a purlin provides interior vertical support to the rafter.

Two options exist to resist the thrust caused by the mutual interaction of the rafter pair at the ridge: (1) the trust is resisted at the eave support (Figures 1(f) and 4(b)); or (2) the purlin resists the thrust (Figures 1(g) and 4(c)). These two options are also statically indeterminate, each with one redundant reaction. The analyses were performed with methods similar to that for the collar-strut system, including only strain energy due to bending, while ignoring that due to axial and shear deformations. Reactions are given in Equations (7) and (8).

1. Thrust at the eave support (Figure 4(b)):

$$A_x = B_x = \frac{wL}{\tan\theta}\left[\frac{\alpha^2 - 5\alpha + 3}{8(1-\alpha)}\right], \qquad A_y = wL\left[\frac{-7\alpha^2 + 7\alpha - 1}{8\alpha(1-\alpha)}\right]$$

$$C_y = wL\left[\frac{-\alpha^2 + \alpha + 1}{8\alpha(1-\alpha)}\right] \qquad \text{Eq. (7)}$$

2. Thrust at the purlin (Figure 4(c)):

$$A_y = wL\left[\frac{\alpha^2 + 3\alpha - 1}{8\alpha}\right], \qquad B_x = C_x = \frac{wL}{\tan\theta}\left[\frac{\alpha^2 - 5\alpha + 3}{8(1-\alpha)}\right]$$

$$C_y = wL\left[\frac{-\alpha^2 + 5\alpha + 1}{8\alpha}\right] \qquad \text{Eq. (8)}$$

For both options, expressions for shear force and bending moment in the rafter are identical to those given in Equations (6) for the collar-strut supported rafter. Clearly then, the moment diagrams must display the same sensitivity to span factor α as shown in Figure 5.

The axial force in the eave-and-purlin-supported rafter depends upon the location of thrust restraint and is discontinuous across attachment point C. For thrust restraint provided by the eave (Figure 4(b)), axial force is given by Equations (9a), where N_1 applies for $0 \le x_1 \le \alpha L$, and N_2 applies for $0 \le x_2 \le (1-\alpha)L$.

$$N_1 = w\left[x_1 - L\left(\frac{-7\alpha^2 + 7\alpha - 1}{8\alpha(1-\alpha)} + \frac{\alpha^2 - 5\alpha + 3}{8(1-\alpha)\tan^2\theta}\right)\right]\sin\theta$$

$$N_2 = -w\left[x_2 + L\left(\frac{\alpha^2 - 5\alpha + 3}{8(1-\alpha)\tan^2\theta}\right)\right]\sin\theta \qquad \text{Eq. (9a)}$$

For thrust restraint provided by the purlin (Figure 4(c)), axial force is given by Equations (9b), where again N_1 applies for $0 \le x_1 \le \alpha L$, and N_2 applies for $0 \le x_2 \le (1-\alpha)L$.

$$N_1 = w\left[x_1 - L\left(\frac{\alpha^2 + 3\alpha - 1}{8\alpha}\right)\right]\sin\theta, \quad N_2 = -w\left[x_2 + L\left(\frac{\alpha^2 - 5\alpha + 3}{8(1-\alpha)\tan^2\theta}\right)\right]\sin\theta \qquad \text{Eq. (9b)}$$

Not surprisingly, the axial force N_2 in the rafter segment between the purlin and the ridge is the same for the two options regardless of the location of thrust restraint, but it differs for the segment between the eave and the purlin, as given by the two expressions for N_1.

Behavior Comparison

To demonstrate the relative effects of the various means of rafter support discussed here, the analysis equations are evaluated for the following parameters:

$$L = 15 \text{ ft}, \quad w = 200 \text{ lb/ft}, \quad \theta = 30°, \text{ and } \alpha = 0.6 \text{ (where applicable).}$$

The support reactions, maximum internal actions, and a moment scale factor are reported in Table 1 for the seven rafter systems. For maximum axial force, both maximum tension N_T and maximum compression N_C forces are reported. The locations of maximum internal actions are not identified, but they can generally be determined by inspection of the reactions or the moment diagrams in Figures 3 and 5.

Rafters act predominately in bending, so bending moment and deflection normally control their design. As such, the moment scale factor is used to facilitate a rapid comparison of the effect of the various support configurations on maximum bending moment. The moment scale factor is the ratio of the maximum moment in the rafter for a particular support configuration to that for eave and ridge support (Figure 1(a)), in which the maximum moment is $wL^2/8$.

Examination of these values reveals that the various support options fall into three different groups. The first group includes eave and ridge support (Figure 1(a)), eave-only support (Figure 1(b)), and eave support with a tie (Figure 1(c)). In this group maximum shear force and bending moment are identical from option to option. Results vary from one option to another only in support reactions and axial force. Hence selection of one of these three support options over any other in the group will be governed by considerations other than maximum bending moment, such as the ability to support a ridge beam, the ability of the eave plate to resist thrust, and headroom requirements.

The second group contains just the collar-tie option (Figure 1(d)), which exhibits a substantial bending moment penalty. It is likely that this option is feasible only for lightly loaded and short-span roof systems. Note again that the tension connection between the collar tie and the rafter may involve joinery that reduces the cross section of the rafter and hence reduces its flexural capacity at the location of maximum bending moment.

The third group contains the statically indeterminate options (Figures 1(e) - (g)), each of which behaves similarly to a two-span continuous beam. For this group the moment scale factor varies from a minimum value of 0.25 for $\alpha = 0.5$ to 0.37 for $\alpha = 0.3$ and $\alpha = 0.7$; values of α outside the range ($0.3 \leq \alpha \leq 0.7$) are not considered practical. The choice of the collar strut (Figure 1(e)) versus purlin support (Figures 1(f) and (g)) will depend on the tradeoff between reduced headroom with the collar strut compared to the need for additional posts and the relatively large purlin. The high thrust at the eave in the collar-strut option is also a consideration.

Table 1 – Comparison of Analysis Results
L = 15 ft, w = 200 lb/ft, θ = 30°, α = 0.6 (where applicable)

Support location	Reactions (lb)	Maximum internal actions (lb, ft-lb)	Moment scale factor
Eave & ridge	$A_x = 0$ $A_y = 1{,}500$ $B_y = 1{,}500$	$N_T = 750$ $N_C = 750$ $V = 1{,}299$ $M = 5{,}625$	1.00
Eave only and Eave with tie	$A_x = 2{,}598$ $A_y = 3{,}000$ $B_x = 2{,}598$	$N_C = 3{,}750$ $V = 1{,}299$ $M = 5{,}625$	1.00
Collar tie	$A_y = 3{,}000$ $B_x = 6{,}495$ $C_x = 6{,}495$	$N_C = 6{,}225$ $V = 3{,}248$ $M = 18{,}900$	3.36
Collar strut	$A_x = 3{,}941$ $A_y = 3{,}000$ $B_x = 585$ $C_x = 3{,}356$	$N_C = 4{,}913$ $V = 931$ $M = 1{,}575$	0.28
Eave & purlin (thrust at eave)	$A_x = 585$ $A_y = 1{,}063$ $B_x = 585$ $C_y = 1{,}938$	$N_C = 1{,}106$ $V = 931$ $M = 1{,}575$	0.28
Eave & purlin (thrust at purlin)	$A_y = 725$ $B_x = 585$ $C_x = 585$ $C_y = 2{,}275$	$N_T = 537$ $N_C = 1{,}106$ $V = 931$ $M = 1{,}575$	0.28

Note: N_T = axial tension, N_C = axial compression, V = shear, M = bending moment.

Conclusions

Selection of rafter support systems in timber buildings depends on a variety of considerations. Designers must understand the impacts, both substantial and nuanced, of the various means of support in order to select systems that satisfy the structural performance requirements of the building and are also compatible with aesthetic and serviceability demands. The systems considered here fall into three behavior groups for which the substantial difference affecting design is the bending moment in the rafter. Nuances within each group can be evaluated to select an appropriate support system.

Based on the analysis results presented here, the following conclusions were drawn.

- The bending moments for rafters spanning uninterrupted from eave to ridge (eave and ridge support, eave-only support, eave support with tie; Figures 1(a) - (d)) are all equal, and thus the choice of the rafter system will be made based on other considerations.
- The collar-tied rafter system (Figure 1(d)) produces the largest bending moments in the rafters and also the largest tensile forces in the ties. While this is a popular framing style for architectural reasons, the structural inefficiencies should give the designer pause when making this selection.
- Of the four statically determinate options, only that with the eave and ridge support (Figure 1(a)) can resist progressive collapse of the rafter pair, should one rafter fail. The two indeterminate options that include purlin support (Figures 1(f) and (g)) also can resist progressive collapse. All other options must rely on supplemental systems to provide any load capacity after failure of a single rafter in the pair.
- Collar struts (Figure 1(e)) greatly reduce the bending moment in rafters without the use of purlin supports but also require a stiff and strong connection at the mating eave. Commonly, collar-strut rafter systems are practical only when there are nearby floor system ties or stiff exterior walls to resist the rafter thrust.
- Purlin-supported rafter systems (Figures 1(f) and (g)) help control bending moment in the rafter while maintaining generous headroom under the roof. However, additional interior framing is required to support the purlins.

References

[1] Schmidt, R. J. and Miller, J. F. (2012) Considerations for Design of Rafters in Timber Buildings, *Practice Periodical on Structural Design and Construction*, ASCE, v 17, n 3, pp127-132, August

[2] Boresi, A. P. and Schmidt, R. J. (2003) *Advanced Mechanics of Materials*, 6E, John Wiley & Sons

[3] AWC (2007) *Design Aid No. 6 – Beam Design Formulas with Shear and Moment Diagrams* American Wood Council, 1111 19th St. NW, Suite 800, Washington, DC 20036

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